

Fragile systems: A hidden variable theory for quantum mechanics

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The formalism of Quantum Mechanics is derived from the application of Bayesian probability theory to “fragile” systems, i.e. systems that are perturbed by the act of measurement. Complex Hilbert spaces, non-commuting operators and the trace rule for expectations all arise naturally from the use of linear algebra to solve integral equations involving classical probabilities over hidden variables. We comment on the case of non-local hidden variables, where violations of Bell’s theorem can be produced, as well as the non-fragile limit of the theory, where all measurements are commutative and the theory becomes analogous to classical statistical mechanics.

Introduction. —Quantum mechanics has been a controversial issue in physics because of its non-local phenomena. In their seminal paper in 1935, Einstein, Podolsky and Rosen [1] asserted that quantum mechanics was incomplete, because of the impossibility of predicting complementary quantities such as position and velocity of a particle at the same time. Moreover, quantum entanglement introduced the notion of non-locality, which was later discussed by means of Bell’s theorem [2]. Entangled systems manifest non-classical correlations between outcomes performed on physical systems that are far apart, but such that they have interacted in the past.

As a consequence of experimental violations of Bell’s inequality [3] it was concluded that quantum mechanics, if it is to be formulated in terms of hidden variables, has to be a non-local theory. In order to attempt to fill this void in the understanding of the foundations of quantum mechanics, a number of hidden-variables theories [4] have been proposed.

In this letter, we will formulate a theory of fragile systems based on non-local hidden variables derived from the application of Bayesian probability. We will recover the formalism of quantum theory from first principles, in particular, we will obtain that,

- The states after a measurement correspond to fixed points of a linear transformation.
- This transformation leads to an eigenvalue equation involving a linear operator in Hilbert space.
- The time evolution of an arbitrary state is described by a linear unitary operator.
- Expectations are given by the trace rule of the density matrix formalism.

Fragile systems. —In simple terms, a fragile system is one that is affected by the act of observation (measurement). This distinguishes it from a non-fragile (classical) system, which is not modified upon observation.

Because any system (being fragile or not) possesses information, we will think of a system as a “black box” that can be found in different **internal states**, to be denoted by λ . In general λ contains many degrees of freedom, but we will not make use of that inner structure here. The internal state λ contains all the information necessary to describe any aspect of the system.

We will consider a system with several real-valued, discrete observables A, B, C, \dots . For instance, the observable A may yield a value given by a real function $R_A(\lambda) \in \{a_1, \dots, a_N\}$. In this case, the statement $a_k = R_A(\lambda)$ means that a measurement of A when the system is found in its internal state λ produced the value a_k .

The crucial difference between a fragile system and a non-fragile one is that, in a fragile system, **access to the internal state λ is impossible**, because it is precisely this internal state which is modified by the act of measurement. We cannot, therefore, assume that we can evaluate R_A on the internal state λ to obtain the outcome of the measurement. As the modification of the state λ depends on the details of the environment doing the measurement (which we do not know or control with accuracy), the outcome of a measurement is unavoidably stochastic, and a mathematical formulation requires probability theory.

In summary, a fragile system is one that (a) modifies itself when it is measured, (b) when it is measured, the system remains in one state and (c) its measurable properties have finite outcomes.

Probability theory. —As we do not have exact knowledge of the internal state λ , we can only assign a probability distribution over it, $P(\lambda|S)$, in **our state of knowledge S** . Unlike non-fragile systems, in a fragile system there is no state of knowledge \mathcal{I} represented with an infinitely sharp peak, $P(\lambda|\mathcal{I}) = \delta(\lambda - \lambda_0)$. Neither can we know the exact modification that a measurement will do on the internal state λ , thus for an observable A we can only assign a transition probability $P(\lambda'|\lambda, A)$ of the final internal state λ' given the initial state λ and that a

measurement of A has occurred.

By the application of the marginalization rule of probability [5], we see that if we are in a state of knowledge S before a measurement of A is made, after the measurement the new state of knowledge S' will be given by

$$P(\lambda'|S') = \int d\lambda P(\lambda|S)P(\lambda'|\lambda, A) = P(\lambda'|S, A), \quad (1)$$

so that $S' = S \wedge A$. In the particular case of a non-fragile system, the internal state λ is not modified by the measurement of A , and therefore $P(\lambda'|\lambda, A) = \delta(\lambda' - \lambda)$, and we have thus $P(\lambda|S') = P(\lambda|S)$.

Fixed points of the transformation. —Consider the situation after a measurement of A yields the value a_k . Bayes' theorem [5] tells us that our state of knowledge must agree with the probability

$$P(\lambda|a_k) = \frac{\delta(R_A(\lambda), a_k)P(\lambda|\mathcal{I}_0)}{P(a_k|\mathcal{I}_0)}, \quad (2)$$

with

$$P(a_k|\mathcal{I}_0) = \int d\lambda P(\lambda|\mathcal{I}_0)\delta(R_A(\lambda), a_k). \quad (3)$$

This implies our state is one of complete knowledge of R_A . We will postulate that the prior probability of the internal states $P(\lambda|\mathcal{I}_0)$ is flat, and then Eq. 2 reduces to

$$P(\lambda|a_k) = \frac{\delta(R_A(\lambda), a_k)}{\Omega_A(a_k)}, \quad (4)$$

where $\Omega_A(a_k) = \int d\lambda \delta(R_A(\lambda), a_k)$ is the density of internal states with given value of R_A . The fact that $P(\lambda|a_k)$ forbids all values of λ with $R_A \neq a_k$ implies that two consecutive measurements of the same observable A , without any perturbation in between, will yield the same outcome a_k . From this it follows that the state of knowledge after a measurement must be a fixed point of the transformation $S \rightarrow S'$ given by Eq. 1. That is, if $g_k(\lambda) = P(\lambda|a_k)$ is the state of knowledge after obtaining the outcome a_k in the measurement of A , we have

$$g_k(\lambda') = \int d\lambda g_k(\lambda)P(\lambda'|\lambda, A). \quad (5)$$

Obviously, this is always the case for a non-fragile system, as $P(\lambda'|\lambda, A) = \delta(\lambda' - \lambda)$.

Representation in terms of a complete basis. —We can construct a complete, orthonormal basis $\{\phi_1(\lambda), \dots, \phi_n(\lambda)\}$ for the probabilities $P(\lambda|S)$. Using the marginalization rule,

$$P(\lambda|S) = \sum_{i=1}^N P(\lambda|a_i)P(a_i|S) = \sum_{i=1}^N \frac{\delta(R_A(\lambda), a_i)}{\Omega_A(a_i)} P(a_i|S). \quad (6)$$

Now define the basis functions

$$\phi_i(\lambda) = \frac{\delta(R_A(\lambda), a_i)}{\sqrt{\Omega_A(a_i)}} \quad (7)$$

such that

$$P(\lambda|S) = \sum_{i=1}^N v_i \phi_i(\lambda). \quad (8)$$

This fixes the coefficients $v_i = P(a_i|S)/\sqrt{\Omega_A(a_i)}$. Because the function $R_A(\lambda)$ is single-valued, $\phi_i(\lambda)\phi_j(\lambda) = 0$ for any λ if $i \neq j$. Furthermore, $\phi_i(\lambda)^2 = P(\lambda|a_i)$, so the basis is orthonormal,

$$\int d\lambda \phi_i(\lambda)\phi_j(\lambda) = \delta_{ij}. \quad (9)$$

Expanding also $P(\lambda'|S')$ in terms of this basis as

$$P(\lambda'|S') = \sum_{j=1}^n w_j \phi_j(\lambda'), \quad (10)$$

we can represent the states of knowledge S and S' by the vectors $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$, respectively, and we can write Eq. 1 as

$$\sum_{j=1}^n w_j \phi_j(\lambda') = \int d\lambda \sum_{i=1}^n v_i \phi_i(\lambda) P(\lambda'|\lambda, A). \quad (11)$$

Multiplying both sides by $\phi_k(\lambda')$ and integrating over λ' , we have

$$\sum_{j=1}^n w_j \int d\lambda' \phi_j(\lambda') \phi_k(\lambda') = \sum_{i=1}^n v_i \int d\lambda d\lambda' \phi_i(\lambda) P(\lambda'|\lambda, A) \phi_k(\lambda'). \quad (12)$$

Now, using the orthonormality condition (Eq. 9),

$$\sum_{j=1}^n w_j \int d\lambda' \phi_j(\lambda') \phi_k(\lambda') = w_k, \quad (13)$$

so we write

$$w_k = \sum_{i=1}^n v_i \int d\lambda d\lambda' \phi_i(\lambda) P(\lambda'|\lambda, A) \phi_k(\lambda'). \quad (14)$$

Defining the matrix \mathbb{T} with elements

$$T_{ij} = \int d\lambda d\lambda' \phi_i(\lambda') P(\lambda'|\lambda, A) \phi_j(\lambda). \quad (15)$$

we can finally write Eq. 1 as a linear transformation,

$$\mathbf{w}_k = \sum_{i=1}^n T_{ki} v_i, \quad (16)$$

equivalent to $\mathbf{w} = \mathbb{T} \cdot \mathbf{v}$. The fixed points of the transformation, namely $g_k(\lambda)$, are now encoded as eigenvectors \mathbf{u}_k (with eigenvalue 1) such that $\mathbf{u}_k = \mathbb{T} \cdot \mathbf{u}_k$.

The matrix \mathbb{T} is the transformation which allows us to obtain the fixed points of the system. On the other hand, we see that we can obtain an analogous operator \mathbb{A} leading to the necessary transformation to obtain also the eigenvalues of the system. For this we instead use

$$Q(\lambda') = \int d\lambda R_A(\lambda) P(\lambda|S) P(\lambda'|\lambda, A), \quad (17)$$

such that for $S = a_k$, $P(\lambda|S) = P(\lambda|a_k) = g_k(\lambda)$. According to Eq. 4, λ has zero probability if $R_A(\lambda) \neq a_k$. Considering this, we see that

$$Q(\lambda') = a_k \int d\lambda g_k(\lambda) P(\lambda'|\lambda, A) = a_k g_k(\lambda'), \quad (18)$$

where the second equality holds because of Equation 5. Finally,

$$a_k g_k(\lambda') = \int d\lambda R_A(\lambda) g_k(\lambda) P(\lambda'|\lambda, A) \quad (19)$$

which, after performing the same basis expansion given by Eqs. 8 and 10, becomes the eigenvalue problem

$$a_i \mathbf{u}_i = \mathbb{A} \cdot \mathbf{u}_i. \quad (20)$$

The matrix elements A_{ij} are given by

$$A_{ij} = \int d\lambda d\lambda' R_A(\lambda) \phi_i(\lambda') P(\lambda'|\lambda, A) \phi_j(\lambda). \quad (21)$$

These matrix elements A_{ij} are real numbers, because the function $R_A(\lambda)$ and the basis functions $\{\phi_i(\lambda)\}$ are real. However, in general it can be more convenient to express the eigenvalue problem in an arbitrary, complex basis $\{\psi_i(\lambda)\}$, so that

$$\mathbb{A} = \sum_{i=1}^N a_i \mathbf{u}_i \otimes \mathbf{u}_i \rightarrow \sum_{i=1}^N a_i \mathbf{c}_i \otimes \mathbf{c}_i^*. \quad (22)$$

with \mathbf{c}_k a complex vector of dimension N , namely the coefficients of $\phi_k(\lambda)$ in the complex basis $\{\psi_i\}$. In this

complex representation, the matrix \mathbb{T} is unitary, and \mathbb{A} is Hermitian.

Time evolution. —Time evolution behaves also as a linear operator acting on states of knowledge,

$$P(\lambda'|S') = \int d\lambda P(\lambda|S) P(\lambda'|\lambda, \Delta t), \quad (23)$$

with $S' = S \wedge \Delta t$. This operator is represented by the unitary matrix

$$U_{ij}(\Delta t) = \int d\lambda d\lambda' \phi_i(\lambda') P(\lambda'|\lambda, \Delta t) \phi_j(\lambda), \quad (24)$$

such that an arbitrary state evolves as

$$\mathbf{w}(t + \Delta t) = \mathbb{U}(\Delta t) \cdot \mathbf{w}(t). \quad (25)$$

In quantum mechanics, this *propagator* $\mathbb{U}(\Delta t)$ is given in terms of the Hamiltonian of the system as $\mathbb{U}(\Delta t) = \exp(-i\mathbb{H}\Delta t/\hbar)$.

Density matrix formalism. —In an arbitrary state of knowledge S we can write the expectation value of the measurement A as

$$\langle R_A \rangle_S = \int d\lambda R_A(\lambda) \sum_{i=1}^N P(\lambda|a_i) P(a_i|S) = \sum_{i=1}^N a_i P(a_i|S). \quad (26)$$

Now, recognizing that the $\{a_i\}$ are the eigenvalues of the matrix \mathbb{A} , as given by Eq. 20, with corresponding eigenvectors \mathbf{u}_i , we can write them as $a_i = (\mathbf{u}_i)^T \mathbb{A} \mathbf{u}_i$. These are known as quadratic forms in Statistics [6]. Then, the expectation in state S is

$$\langle R_A \rangle_S = \sum_{i=1}^N p_i (\mathbf{u}_i)^T \mathbb{A} \mathbf{u}_i = \sum_{l,m} \rho_{ml} A_{lm} = \text{Tr}(\rho \mathbb{A}), \quad (27)$$

where the density matrix ρ associated to the state of knowledge S is defined as

$$\rho = \sum_{i=1}^N p_i \mathbf{u}_i \otimes \mathbf{u}_i, \quad (28)$$

with $p_i = P(a_i|S)$ and \otimes the Kronecker product, $\mathbf{v} \otimes \mathbf{w} = \mathbf{vw}^T$. This is a properly defined density matrix because the p_i are probabilities of discrete propositions, non-negative and adding up to 1. We can see that every system where we can write the expectation values of its properties in terms of the quadratic forms in Eq. (27) and leading to non-commutative operators, can be considered as a fragile system. It is straightforward to show, using Eq. 25 on a particular basis expansion, that the density

operator follows a von Neumann evolution, $\rho(t + \Delta t) = \mathbb{U}(\Delta t)\rho(t)\mathbb{U}^\dagger(\Delta t)$.

Bell's Theorem —In the present theory, the internal variables λ play the role of a set of hidden variables for quantum mechanics. Consider two systems A and B and let a and a' detector settings on the system A , b and b' on the system B . The expectation values of the possible measurements for A and B with arbitrary settings a and b are given by

$$\begin{aligned}\langle R_A \rangle_{a,S} &= \int d\lambda R_A(\lambda; a) P(\lambda|a, S), \\ \langle R_B \rangle_{b,S} &= \int d\lambda R_B(\lambda; b) P(\lambda|b, S).\end{aligned}\quad (29)$$

Taking the joint expected value for the system A and B for the settings a and b gives,

$$\langle R_A R_B \rangle_{a,b,S} = \int d\lambda R_A(\lambda; a) R_B(\lambda; b) P(\lambda|a, b, S), \quad (30)$$

which in general is different from $\langle R_A \rangle_{a,S} \langle R_B \rangle_{b,S}$ because of the existence of correlations between the two systems. Bell's inequality, which in our notation reads

$$\left| \langle R_A R_B \rangle_{a,b,S} - \langle R_A R_B \rangle_{a,a',S} + \langle R_A R_B \rangle_{b,b',S} + \langle R_A R_B \rangle_{a',b',S} \right| \leq 2. \quad (31)$$

was applied in quantum mechanics in order to put some restrictions in its postulates on the possible hidden variables theories [7–9].

In the case where the probability distribution $P(\lambda|a, b, S)$ for the systems A and B is not separable we have that every observable value depends on λ . We have then a non-local hidden variable theory by construction. Therefore the correlations of the systems A and B are non-local.

This shows that it is possible to have violations of Bell's inequality in a macroscopic system if the corresponding hidden variables are non-local, and an interesting example of this behavior is given by Aerts[10].

Conclusions. —We have shown that fragile systems with discrete properties can be analyzed in terms of genuine quantum mechanics, complete with non-commuting operators and a density matrix formalism in complex Hilbert space. This not only gives a strong probabilistic justification for the fact that Nature itself is quantum mechanical, but it also opens the possibility of employing the structure of quantum mechanics as an inference tool in problems involving fragile systems outside physics, in areas such as biology, data analysis [11], dynamical systems [12] among others. For instance, it would be possible to apply this tool to biological models, under the perspective of autopoietic systems[13] having self-modifying properties, as we have shown that indeed every system able to modify itself can be considered, under this formulation, as a fragile system. There is also the interesting possibility of applying our results as a formal justification of the recent idea of quantum cognition [14] in which the

object of study is human logic and human decisions.

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